Bistable Traveling Waves for Monotone Discrete-time Recursive Systems

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(Based on the joint works with Jian Fang and Yuxiang Zhang)

Outline

1. Bistable waves
2. The general theory
3. A competition model

Fife and Mcleod's result, 1977,81

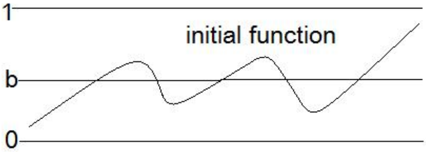
The scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + u(1-u)(u-b), \quad x \in \mathbb{R}, t \geq 0, b \in (0,1) \qquad (1.1)$$

admits a unique (up to translation) monotone bistable wave $U(x+ct)$, and there exists a constant $\mu > 0$ such that the solution $u(t,x;\psi)$ with $u(0)=\psi$ satisfies

$$\|u(t,x;\psi) - U(x+ct+s_\psi)\| \leq C_\psi e^{-\mu t}$$

for some constant s_ψ and $C_\psi > 0$ provided that the initial function ψ is bounded and uniformly continuous, and has the property:



A spruce budworm model

Ludwig, Jones and Holling (1978, J. Anim. Ecol.) presented a budworm population model

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2},$$

where r_B is the linear birth rate of the budworm, K_B is the carrying capacity, which is related to the density of foliage available on the trees, and the term $\frac{BN^2}{A^2+N^2}$ with $A, B > 0$ represents predation, generally by birds.

Field observation shows that there are three possible positive steady states for the population. The smallest steady state α^- is the refuge equilibrium, while α^+ is the outbreak equilibrium. From a pest control point of view, we should try to keep the population at a refuge state rather than allow it to reach an outbreak situation.

Evolution systems

- Lui (1983,JMB) studied the integrodifference equation

$$u_{n+1}(x) = \int_{\mathbb{R}} k(x-y)g(u_n(y))dy, \quad x \in \mathbb{R}. \tag{1.2}$$

- Schaaf (1987, TAMS) obtained the existence of bistable waves for the time-delayed reaction-diffusion equation

$$u_t = u_{xx} + f(u, u(t-\tau)), \quad x \in \mathbb{R}. \tag{1.3}$$

  Smith and Zhao (2000, SIMA) proved the global stability.
- Bates, Fife, Ren and Wang (1997, Arch. Ration. Mech. Anal.) established the existence and stability for the phase transition model

$$u_t = J * u - u + f(u), \tag{1.4}$$

  where $J \in C^1(\mathbb{R}, \mathbb{R})$ with $\int_{\mathbb{R}} J(x)dx = 1$.
  Coville (2007, preprint) and Yagisita (2009, Publ. RIAM, Kyoto Univ.) did further extensions.

More works

- Chen (1999, Adv. Diff. Eqs.) established a general result on the existence and stability for the nonlocal evolution equation

$$u_t(t,x) = \mathcal{A}[u(t,\cdot)](x), \quad x \in \mathbb{R}, \tag{1.5}$$

where $\mathcal{A}$ generates a semiflow on $L^\infty(\mathbb{R})$.

Nonlocal dispersal equations
Reaction and dffusion systems
Time-delayed reaction-diffusion equations
Lattice equations
Periodic and almost periodic equations
Evolution equations in a periodic habitat

Our purpose

We propose to establish the general theory on the existence of **bistable waves** for monotone evolution systems, and then apply it to some important spatial models such as two species competition systems.

We also present a dynamical systems approach to the **global stability** of such waves.

Notations

Let habitat $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$, and $\mathcal{T} = \mathbb{R}^+$ or $\mathbb{Z}^+$. Assume that $\mathcal{X}$ is a Banach Lattice with the norm $\|\cdot\|_{\mathcal{X}}$ and the positive cone $\mathcal{X}^+$, and $\mathcal{C} := BC(\mathcal{H}, \mathcal{X})$ equipped with the compact open topology. For example, $\mathcal{X} = \mathbb{R}^m$ and $\mathcal{X}^+ = \mathbb{R}^m_+$.

Let $\beta \in Int(\mathcal{X}^+)$ and Q be a map from $\mathcal{C}_\beta$ to $\mathcal{C}_\beta$, where $\mathcal{C}_\beta := [0, \beta]_{\mathcal{C}}$. Let E be the set of fixed points of Q restricted on $\mathcal{X}_\beta := [0, \beta]_{\mathcal{X}}$.

Assumptions

Let 0 and β be in E. We assume that

(A1) (*Translation Invariance*)
$T_y \circ Q[\phi] = Q \circ T_y[\phi], \forall \phi \in \mathcal{C}_\beta, y \in \mathcal{H}$, where
$T_y[\phi](x) = \phi(x - y)$.

(A2) (*Continuity*) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology.

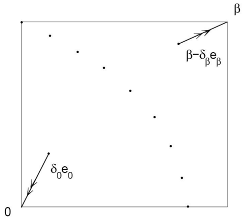
(A3) (*Monotonicity*) Q is order preserving in the sense that $Q[\phi] \geq Q[\psi]$ whenever $\phi \geq \psi$ in $\mathcal{C}_\beta$.

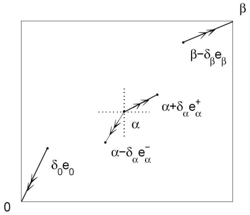
(A4) (*Compactness*) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is compact with respect to the compact open topology.

(A5) (*Bistability*) Fixed points 0 and β are are strongly stable from above and below, respectively, for $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, and the set $E \setminus \{0, \beta\}$ is totally unordered.

A sufficient condition for hypothesis (A5) to hold is:

(A5′) (*Bistability*) $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$ is eventually strongly monotone. Further, for $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, two fixed points 0 and β are strongly stable from above and below, respectively, and each $\alpha \in E \setminus \{0, \beta\}$ (if exists) is strongly unstable from both below and above.

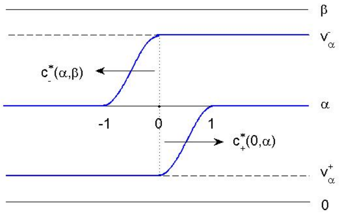




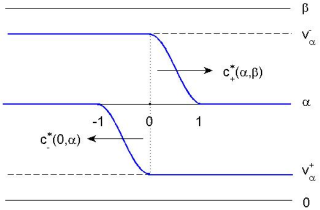
Definition (Spreading speeds in a weak sense)

$$c_-^*(\alpha,\beta) := \sup\{c \in \mathbb{R} : \lim_{n\to\infty, x\geq -cn} Q^n[\phi_\alpha^-](x) = \beta\}.$$

$$c_+^*(0,\alpha) := \sup\{c \in \mathbb{R} : \lim_{n\to\infty, x\leq cn} Q^n[\phi_\alpha^+](x) = 0\}.$$



Similarly, we can define $c_+^*(\alpha, \beta)$ and $c_-^*(0, \alpha)$, respectively.



Monostable systems

For the abstract theory of spreading speeds and traveling waves in the monostable case, we refer to the following papers and references therein.

Weinberger (1982, SIAM J. Math. Anal.): maps on $C(\mathcal{H}, \mathbb{R})$.
Lui (1989,Math. Biosci.): maps on $C(\mathcal{H}, \mathbb{R}^m)$.
Weinberger (2002, J. Math. Biol.): maps on $C(\mathcal{H}, \mathbb{R}^m)$ in a periodic habitat.
Li, Weinberger and Lewis (2005, Math. Biosci.): reaction-diffusion systems. "positive diffusion coefficients"
Liang and Zhao (2007, Commun. Pure Appl. Math.): semiflows on $C(M \times \mathcal{H}, \mathbb{R}^m)$. Here M is a compact metric space, e.g., $M = [-\tau, 0]$ for time-delayed equations, and $M = \bar{\Omega}$ for parabolic equations in a cylinder.
Liang and Zhao (2010, J. Functional Analysis): monotone semiflows on $\mathcal{K} \subset C(\mathcal{H}, X)$ with X being a Banach lattice and $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$.

The counter-propagation

Now we present the last assumption on Q:

(A6) (*Counter-propagation*) For each $\alpha \in E \setminus \{0, \beta\}$, $c_-^*(\alpha, \beta) + c_+^*(0, \alpha) > 0$.

Assumption (A6) assures that two initial functions in the above figure will eventually spread oppositely although one of these two speeds may be negative. If we consider the non-increasing traveling waves, then (A6) should be stated as $c_+^*(\alpha, \beta) + c_-^*(0, \alpha) > 0$.

Here we consider the case where $\mathcal{H} = \mathbb{R}$ and $\mathcal{T} = \mathbb{Z}^+$.

Definition

$\psi(x + cn)$ with $\psi \in \mathcal{C}$ is said to be a traveling wave with speed $c \in \mathbb{R}$ of the discrete-time semiflow $\{Q^n\}_{n\geq 0}$ if

$$Q^n[\psi](x) = \psi(x + cn), \quad \forall x \in \mathbb{R}, n \geq 0.$$

Theorem

Assume that Q satisfies (A1)-(A6). Then there exists $c \in \mathbb{R}$ such that the discrete-time semiflow $\{Q^n\}_{n\geq 1}$ admits a non-decreasing traveling wave with speed c and connecting 0 to β.

Extensions

(1) The continuous habitat $\mathcal{H} = \mathbb{R}$, and $\mathcal{T} = \mathbb{R}^+$.

(2) The discrete habitat $\mathcal{H} = \mathbb{Z}$, either $\mathcal{T} = \mathbb{Z}^+$, or $\mathcal{T} = \mathbb{R}^+$.

(3) Time-periodic systems.

(4) A periodic habitat.

(5) Weak compactness assumptions.

A convergence theorem

To study the stability of bistable waves, we need the following result which comes from Zhao's 2003 book.

Lemma

Let U be a closed and order convex subset of an ordered Banach space $\mathcal{X}$ with nonempty positive cone, and $f : U \to U$ continuous and monotone. Assume that there exists a monotone homeomorphism h from $[0,1]$ onto a subset of U such that

(i) *For each $s \in [0,1], h(s)$ is a stable fixed point for $f : U \to U$;*

(ii) *Each forward orbit of f on $[h(0), h(1)]_{\mathcal{X}}$ is precompact;*

(iii) *If $\omega(x) > h(s_0)$ for some $s_0 \in [0,1)$ and $x \in [h(0), h(1)]_{\mathcal{X}}$, then there exists $s_1 \in (s_0, 1)$ such that $\omega(x) > h(s_1)$.*

Then for any precompact orbit $\gamma^+(y)$ of f in U with $\omega(y) \bigcap [h(0), h(1)]_{\mathcal{X}} \neq \emptyset$, there exists $s^* \in [0,1]$ such that $\omega(y) = h(s^*)$.

The model

Consider the discrete-time two species **competition** model:

$$\begin{aligned} p_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1q_n(x-y))} k_1(y)dy, \\ q_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_2)q_n(x-y)}{1+r_2(q_n(x-y)+a_2p_n(x-y))} k_2(y)dy. \end{aligned} \tag{3.1}$$

Here $p_n(x)$ and $q_n(x)$ denote the population densities of two species at time n and position x, respectively; $k_i(y)$ represents the dispersal kernel of two species and

$$\int_{\mathbb{R}} k_i(y)dy = 1, \quad \int_{\mathbb{R}} e^{\alpha y} k_i(y)dy < \infty, \ \forall \alpha \in \mathbb{R}, i = 1, 2.$$

Assume that all parameters are positive constants and the kernel k_i has the symmetric property $k_i(-y) = k_i(y)$, which implies that the dispersal is isotropic and that the growth and dispersal properties are the same at each point.

It is easy to see that the change of variables

$$u_n = p_n, \;\; v_n = 1 - q_n$$

converts system (3.1) into the following **cooperative** system:

$$\begin{aligned} u_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_1)u_n(x-y)}{1+r_1(u_n(x-y)+a_1(1-v_n(x-y)))} k_1(y)dy, \\ v_{n+1}(x) &= \int_{\mathbb{R}} \frac{a_2 r_2 u_n(x-y)+v_n(x-y)}{1+r_2((1-v_n(x-y))+a_2 u_n(x-y))} k_2(y)dy, \end{aligned} \tag{3.2}$$

which is order-preserving in the relevant range $0 \le u_n \le 1, 0 \le v_n \le 1$. Note that system (3.1) has four possible constant equilibria: $(0,0), (0,1)$, $(1,0)$, and (p^*, q^*), where

$$p^* = \frac{1-a_1}{1-a_1 a_2}, \;\; q^* = \frac{1-a_2}{1-a_1 a_2}.$$

Thus, the cooperative system (3.2) has four equilibria: $E^0 = (0,1)$, $E^1 = (0,0)$, $E^2 = (1,1)$, and $E^3 = (u^*, v^*)$, where $u^* = p^*, v^* = 1 - q^*$. Note that the positive coexistence equilibrium exists if and only if $(1-a_1)(1-a_2) > 0$. For the spatially homogeneous system associated with (3.1):

$$\begin{aligned} p_{n+1} &= \frac{(1+r_1)p_n}{1+r_1(p_n+a_1q_n)}, \\ q_{n+1} &= \frac{(1+r_2)q_n}{1+r_2(q_n+a_2p_n)}, \end{aligned} \tag{3.3}$$

Cushing et al. (2004, JDEA) gave a complete classification of its global dynamics.

Bistable case

In the case where $a_1, a_2 \in (1, +\infty)$, the equilibrium (p^*, q^*) is a saddle, (0,1) and (1,0) are stable, and (0,0) is unstable for the system (3.3). Further, there exists a separatrix Γ such that all orbits of system (3.3) below Γ converges to (1,0), while all orbits of system (3.3) above Γ converges to (0,1).

Here we are interested in the existence of **bistable travelling waves** connecting (0,1) and (1,0), and their **global stability** with phase shift.

Define an operator $Q = (Q_1, Q_2)$ on $\mathcal{C}$ by

$$Q_1[u,v](x) = \int_{\mathbb{R}} \frac{(1+r_1)u(x-y)}{1+r_1(u(x-y)+a_1(1-v(x-y)))} k_1(y)dy,$$

$$Q_2[u,v](x) = \int_{\mathbb{R}} \frac{a_2 r_2 u(x-y)+v(x-y)}{1+r_2((1-v(x-y))+a_2 u(x-y))} k_2(y)dy.$$

Then system (3.2) can be expressed as

$$U_{n+1}(x) = Q[U_n](x),\ U_n := (u_n, v_n),\ n \geq 0.$$

Lemma

The map Q satisfies (A1)-(A6) with $\beta = E^2$ and $E = \{E^0, E^1, E^2, E^3\}$.

Existence of traveling waves

By the general theory in section 2, we have the following result.

Theorem

Let all parameters be positive and $a_1, a_2 \in (1, \infty)$. *Then there exists* $c \in \mathbb{R}$ *such that the cooperative system* (3.2), *which obtained by making substitution* $u_n = p_n, v_n = 1 - q_n$ *in model* (3.1), *has a nondecreasing traveling wave* $\varphi(x - cn)$ *with speed* c *and connecting two stable equilibria* $E^1 = (0, 0)$ *and* $E^2 = (1, 1)$.

Next we study the **global stability and uniqueness** of bistable traveling waves for system (3.2).

Let $\varphi(x-cn) = (\varphi_1(x-cn), \varphi_2(x-cn))$ be a nondecreasing traveling wave solution of (3.2) connecting E^1 to E^2. Letting $z = x - c(n+1)$, we transform (3.2) into the following system

$$\bar{U}_{n+1}(z) = T_{-c} \circ Q[\bar{U}_n](z), \;\; n \geq 0. \tag{3.4}$$

Thus, φ is an equilibrium solution of system (3.4), that is, $\varphi = T_{-c} \circ Q[\varphi]$.

In what follows, we denote $\bar{U}_n(z, \psi)$ to be the solution of (3.4) with initial data $\bar{U}_0 = \psi$. Clearly, the solution $U_n(x, \psi)$ of (3.2) with initial data ψ is given by $U_n(x, \psi) = \bar{U}_n(x - cn, \psi)$.

Lemma

(i) *If* $\psi \in \mathcal{C}_{[E^1,E^2]}$ *is nondecreasing and satisfies*

$$\limsup_{\xi\to-\infty}\psi(\xi) \ll E^3 \ll \liminf_{\xi\to\infty}\psi(\xi), \tag{3.5}$$

then for any $\varepsilon > 0$, *there exists* $\tilde{z} = \tilde{z}(\varepsilon, \psi) > 0$ *such that* $\varphi(z-\tilde{z}) - \vec{\varepsilon} \leq \bar{U}_0(z,\psi) \leq \varphi(z+\tilde{z}) + \vec{\varepsilon}$.

(ii) *If the kernel* $k_i, i = 1, 2,$ *has a compact support, then for any* $\varepsilon > 0$ *and* $\psi \in \mathcal{C}_{[E^1,E^2]}$ *satisfying* (3.5), *there exist* $\tilde{z} = \tilde{z}(\varepsilon, \psi) > 0$ *and a large time* $n_0 \in \mathbb{N}^+$ *such that* $\varphi(z-\tilde{z}) - \vec{\varepsilon} \leq \bar{U}_{n_0}(z,\psi) \leq \varphi(z+\tilde{z}) + \vec{\varepsilon}$.

Upper and lower solutions

In order to use the method of upper and lower solutions, we first introduce the following concepts.

Definition

A sequence of functions $W_n^+(z) \in C(\mathbb{R}, \mathbb{R}^2), n \geq 0$, is an upper solution of (3.4) if $W_n^+(z)$ satisfies

$$W_{n+1}^+ \geq T_{-c} \circ Q[W_n^+], \quad n \geq 0.$$

A sequence of functions $W_n^-(z) \in C(\mathbb{R}, \mathbb{R}^2), n \geq 0$, is a lower solution of (3.4) if $W_n^-(z)$ satisfies

$$W_{n+1}^- \leq T_{-c} \circ Q[W_n^-], \quad n \geq 0.$$

Lemma

There exist positive number σ and $\varepsilon_0 \in (0,1)$ such that for any $\hat{z}$ and $\varepsilon \in (0, \varepsilon_0)$,

$$W_n^{\pm} = \varphi(z \pm \hat{z} \pm \varepsilon(1 - e^{-\sigma n})) \pm \varepsilon\rho(z \pm \hat{z})e^{-\sigma n}, \ \forall z \in \mathbb{R}, \ n \geq 0$$

are upper and lower solutions of system (3.4)*, respectively.*

Here $\rho(z) : \mathbb{R} \to \mathbb{R}^2$ is a positive nondecreasing map such that $\rho(z) = \rho^+, \forall z \geq z_1 > 0$, and $\rho(z) = \rho^-, \forall z \leq z_2 < 0$, where $z_i, i = 1, 2$, are two fixed real number and $\rho^{\pm} = (\rho_1^{\pm}, \rho_2^{\pm})$ are eigenvectors of $A^{\pm}$ corresponding to the principle eigenvalues satisfying $\vec{0} \ll \rho^- \leq \rho^+ \leq \vec{1}$, and

$$A^- = \begin{pmatrix} \frac{1+r_1}{1+a_1 r_1} & \epsilon_1 \\ \frac{a_2 r_2}{1+r_2} & \frac{1}{1+r_2} \end{pmatrix}, \ A^+ = \begin{pmatrix} \frac{1}{1+r_1} & \frac{a_1 r_1}{1+r_1} \\ \epsilon_1 & \frac{1+r_2}{1+a_2 r_2} \end{pmatrix}.$$

Liapunov stability

Lemma

The wave profile φ is a Liapunov stable equilibrium of (3.4).

Let $\mathcal{D} = BUC(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the usual supreme norm.

Let $\mathcal{D}_+ = \{(\psi_1, \psi_2) \in \mathcal{D} : \psi_i(x) \geq 0, \forall x \in \mathbb{R}, i = 1, 2\}$. Then $\mathcal{D}_+$ is a closed cone of $\mathcal{D}$ and its induced partial ordering makes $\mathcal{D}$ into a Banach lattice.

By the convergence theorem in section 2, we have the following result.

Global stability and uniqueness

Theorem

Let $\varphi(x - cn)$ be a monotone travelling wave solution of system (3.2) and $U_n(x, \psi)$ be the solution of (3.2) with $U_0(\cdot, \psi) = \psi(\cdot) \in \mathcal{D}_{[E^1, E^2]}$. Then the following statements are valid:

(i) *For any nondecreasing $\psi \in \mathcal{D}_{[E^1, E^2]}$ satisfying (3.5), there exists $s_\psi \in \mathbb{R}$ such that $\lim_{n\to\infty} \|U_n(x, t, \psi) - \varphi(x - cn + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$, and any monotone travelling wave solution of system (3.2) connecting E^1 to E^2 is a translation of φ.*

(ii) *If $k_i, i = 1, 2$, has a compact support, then for any $\psi \in \mathcal{D}_{[E^1, E^2]}$ satisfying (3.5), there exists $s_\psi \in \mathbb{R}$ such that $\lim_{n\to\infty} \|U_n(x, t, \psi) - \varphi(x - cn + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$, and any travelling wave solution of system (3.2) connecting E^1 to E^2 is a translation of φ.*

Numerical simulations

We have shown system (3.2) admits a unique monotone bistable travelling wave up to translation, which is globally stable with phase shift. In order to simulate this result, we truncate the infinite domain $\mathbb{R}$ to finite domain $[-L, L]$, where L is sufficiently large. Let $a_1 = 6/5$, $a_2 = 10$, $r_1 = 1/9$, $r_2 = 1/10$, $k_1(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)$, and $k_2(y) = \frac{1}{\sqrt{4\pi}} \exp(-y^2/4)$. The evolution of the solution is shown in Figure 2 for $L = 60$ with the initial condition and the numerical wave profile are shown in Figure 1. We can see, under the given parameters and kernel functions, that the solution rapidly converges to the numerical wave profile, and the **sign** of the spreading speed is **negative**. Note that

$$u_0(x) = \begin{cases} 1/800, & -60 \le x \le -10; \\ 799/800 + 798(x-10)/16000, & -10 \le x \le 10; \\ 799/800, & 10 \le x \le 60. \end{cases}$$

$$v_0(x) = \begin{cases} 1/1000, & -60 \le x \le -10; \\ 899/1000 + 898(x-10)/20000, & -10 \le x \le 10; \\ 899/1000, & 10 \le x \le 60. \end{cases}$$

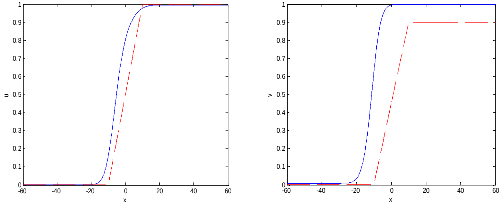


Figure 1. The initial condition and numerical wave profile

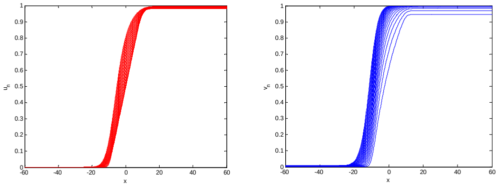


Figure 2. The evolution of u_n and v_n when n=1,2,...,20.

References

We remark that for this two species competition model, it remains an **open problem** to determine the sign of the wave speed c, which tells us which species is the winner of the competition.

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Thank you!